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Ami Harten

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Institute for Computer Applications in Science and Engineering
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Recent Developments in Shock-Capturing Schemes

Ami Harten[1]
School of Mathematical Sciences
Tel-Aviv University
Tel-Aviv ISRAEL
and
Department of Mathematics
University of California
Los Angeles, CA 90024-1555

ABSTRACT

In this paper we review the development of the shock-capturing methodology, paying special attention to the increasing nonlinearity in its design and its relation to interpolation. It is well-known that high-order approximations to a discontinuous function generate spurious oscillations near the discontinuity (Gibbs phenomenon). Unlike standard finite-difference methods which use a fixed stencil, modern shock-capturing schemes use an adaptive stencil which is selected according to the local smoothness of the solution. Near discontinuities this technique automatically switches to one-sided approximations, thus avoiding the use of discontinuous data which brings about spurious oscillations.

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1 Introduction

In this paper, we describe and analyze numerical techniques that are designed to approximate weak solutions of hyperbolic systems of conservation laws in several space dimensions. For sake of exposition, we shall describe these methods as they apply to the pure initial value problem (IVP) for a one-dimensional scalar conservation law

$$u_t + f(u)_x = 0, \quad u(x,0) = u_0(x). \tag{1.1}$$

To further simplify our presentation, we assume that the flux $f(u)$ is a convex function, i.e., $f''(u) > 0$ and that the initial data $u_0(x)$ are piecewise smooth functions which are either periodic or of compact support. Under these assumptions, no matter how smooth u_0 is, the solution $u(x,t)$ of the IVP (1.1) becomes discontinuous at some finite time $t = t_c$. In order to extend the solution for $t > t_c$, we introduce the notion of weak solutions, which satisfy

$$\frac{d}{dt}\int_a^b u\,dx + f(u(b,t)) - f(u(a,t)) = 0 \tag{1.2a}$$

for all $b \geq a$ and $t \geq 0$. Relation (1.2a) implies that $u(x,t)$ satisfies the PDE in (1.1) wherever it is smooth, and the Rankine-Hugoniot jump relation

$$f(u(y+0,t)) - f(u(y-0,t)) = [u(y+0,t) - u(y-0,t)]\frac{dy}{dt} \tag{1.2b}$$

across curves $x = y(t)$ of discontinuity.

It is well-known that weak solutions are not uniquely determined by their initial data. To overcome this difficulty, we consider the IVP (1.1) to be the vanishing viscosity limit $\varepsilon \downarrow 0$ of the parabolic problem

$$(u^\epsilon)_t + f(u^\epsilon)_x = \varepsilon u^\epsilon_{xx} \quad u^\epsilon(x,0) = u_0(x), \tag{1.3a}$$

and identify the unique "physically relevant" weak solution of (1.1) by

$$u = \lim_{\epsilon\downarrow 0} u^\epsilon. \tag{1.3b}$$

The limit solution (1.3) can be characterized by an inequality that the values $u_L = u(y-0,t), u_R = u(y+0,t)$ and $s = dy/dt$ have to satisfy; this inequality is called an entropy condition; admissible discontinuities are called shocks. When $f(u)$ is convex, this inequality is equivalent to Lax's shock condition

$$a(u_L) > s > a(u_R) \tag{1.4}$$

where $a(u) = f'(u)$ is the characteristic speed (see [8] for more details).

We turn now to describe finite difference approximations for the numerical solution of the IVP (1.1). Let v_j^n denote the numerical approximation to $u(x_j, t_n)$ where $x_j = jh, t_n = n\tau$; let $v_h(x,t)$ be a globally defined numerical approximation associated with the discrete values $\{v_j^n\}, \infty < j < \infty, n \geq 0$.

The classical approach to the design of numerical methods for partial differential equations is to obtain a solvable set of equations for $\{v_j^n\}$ by replacing derivatives in the PDE by appropriate discrete approximations. Therefore, there is a conceptual difficulty in applying classical methods to compute solutions which may become discontinuous. Lax and Wendroff [9] overcame this difficulty by considering numerical approximations to the *weak formulation* (1.2a) rather than to the PDE (1.1). For this purpose, they have introduced the notion of schemes in conservation form:

$$v_j^{n+1} = v_j^n - \lambda(\bar{f}_{j+\frac{1}{2}} - \bar{f}_{j-\frac{1}{2}}) \equiv (E_h \cdot v^n)_j; \tag{1.5a}$$

here $\lambda = \tau/h$ and $\bar{f}_{i+\frac{1}{2}}$ denotes

$$\bar{f}_{i+\frac{1}{2}} = f(v_{i-k+1}^n, \ldots, v_{i+k}^n); \tag{1.5b}$$

$\bar{f}(w_1, \ldots, w_{2k})$ is a numerical flux function which is consistent with the flux $f(u)$, in the sense that

$$\bar{f}(u, u, \ldots u) = f(u); \tag{1.5c}$$

E_h denotes the numerical solution operator. Lax and Wendroff proved that if the numerical approximation converges boundedly almost everywhere to some function u, then u is a weak solution of (1.1), i.e., it satisfies the weak formulation (1.2a). Consequently discontinuities in the limit solution automatically satisfy the Rankine-Hugoniot relation (1.2b). We refer to this methodology as shock-capturing (a phrase coined by H. Lomax).

In the following, we list the numerical flux function of various 3-point schemes ($k = 1$ in (1.5b)):

(i) The Lax-Friedrichs scheme [7]

$$\bar{f}(w_1, w_2) = \frac{1}{2}[f(w_1) + f(w_2) - \frac{1}{\lambda}(w_2 - w_1)] \tag{1.6}$$

(ii) Godunov's scheme [1]

$$\bar{f}(w_1, w_2) = f(V(0; w_1, w_2)); \tag{1.7a}$$

here $V(x/t; w_1, w_2)$ denotes the self-similar solution of the IVP (1.1) with the initial data

$$u_0(x) = \begin{cases} w_1 & x < 0 \\ w_2 & x > 0 \end{cases}. \tag{1.7b}$$

(iii) The Cole-Murman scheme [12]:

$$\bar{f}(w_1, w_2) = \frac{1}{2}[f(w_1) + f(w_2) - |\bar{a}(w_1, w_2)|(w_2 - w_1)] \tag{1.8a}$$

where

$$\bar{a}(w_1, w_2) = \begin{cases} \frac{f(w_2) - f(w_1)}{w_2 - w_1} & \text{if } w_1 \neq w_2 \\ a(w_1) & \text{if } w_1 = w_2 \end{cases}. \tag{1.8b}$$

(iv) The Lax-Wendroff scheme [9]:

$$\bar{f}(w_1, w_2) = \frac{1}{2}\{f(w_1) + f(w_2) - \lambda a\left(\frac{w_1 + w_2}{2}\right)[f(w_2) - f(w_1)]\}. \qquad (1.9)$$

Let $E(t)$ denote the evolution operator of the exact solution of (1.1) and let E_h denote the numerical solution operator defined by the RHS of (1.5a). We say that the numerical scheme is r-th order accurate (in a pointwise sense) if its local truncation error satisfies

$$E(\tau) \cdot u - E_h \cdot u = 0(h^{r+1}) \qquad (1.10)$$

for all sufficiently smooth u; here $\tau = 0(h)$. If $r > 0$, we say that the scheme is consistent.

The schemes of Lax-Friedrichs (1.6), Godunov (1.7), and Cole-Murman (1.8) are first order accurate; the scheme of Lax-Wendroff (1.9) is second order accurate.

We remark that the Lax-Wendroff theorem states that if the scheme is convergent, then the limit solution satisfies the weak formulation (1.2b); however, it need not be the entropy solution of the problem (see [4]). It is easy to see that the schemes of Cole-Murman (1.8) and Lax-Wendroff (1.9) admit a stationary "expansion shock" (i.e., $f(u_L) = f(u_R)$ with $a(u_L) < a(u_R)$) as a steady solution. This problem can be easily rectified by adding sufficient numerical dissipation to the scheme (see [11] and [3]).

2 Interpolatory Schemes and Linear Discontinuities

Let us consider the constant coefficient case $f(u) = au, a =$ const. in (1.1), i.e.,

$$u_t + au_x = 0, \quad u(x, 0) = u_0(x), \qquad (2.1a)$$

the solution to which is

$$u(x, t) = u_0(x - at). \qquad (2.1b)$$

In this case the schemes (1.6) - (1.9) take the form

$$v_j^{n+1} = \sum_{\ell=-K^-}^{K^+} C_\ell(\nu) v_{j+\ell}^n \equiv (E_h \cdot v^n)_j \qquad (2.2)$$

where $\nu = \lambda a$ is the CFL number. The coefficients $C_\ell(\nu)$ are independent of the numerical solution v^n; this makes E_h a linear operator.

We say that the numerical scheme E_h is (linearly) stable if

$$\|(E_h)^n\| \leq C \quad \text{for} \quad 0 \leq n\tau \leq T, \quad \tau = O(h). \qquad (2.3a)$$

In the constant coefficient case the scheme is stable if and only if it satisfies von Neumann's condition

$$| \sum_{\ell=-K^-}^{K^+} C_\ell(\nu) \ell^{i\ell\xi} | \leq 1 \quad \text{for all} \quad 0 \leq \xi \leq \pi. \qquad (2.3b)$$

It is easy to see that all the 3 point schemes (1.6) - (1.9) are stable under the CFL condition

$$|\nu| = |\lambda a| \leq 1. \tag{2.3c}$$

The notion of stability (2.3a) is related to convergence through Lax's equivalence theorem, which states that a consistent linear scheme is convergent if and only if it is stable (see [13] for more details).

Let us denote by S_i^r the stencil of $(r+1)$ successive points starting with x_i

$$S_i^r = \{x_i, x_{i+1}, \dots, x_{i+r}\}, \tag{2.4a}$$

let $P(x; S_i^r, u)$ denote the unique polynomial of degree r interpolating the $(r+1)$ values of u on this stencil and let $Q(x; u)$ denote the piecewise polynomial interpolation of u

$$Q(x; u) = P(x; S_{i(j)}^r; u) \quad x_{j-1} \leq x \leq x_j \tag{2.4b}$$

We refer to the numerical scheme

$$v_j^{n+1} = Q(x_j - a\tau; v^n) \tag{2.4c}$$

as interpolatory scheme. Clearly, the interpolatory scheme (2.4) is r-th order accurate. When $Q(x; v^n)$ is the piecewise linear interpolation of v^n (i.e., $r = 1, i(j) = j - 1$ in (2.4b)) then (2.4c) is the first-order accurate upwind scheme; in the constant coefficient case this scheme is identical to those of Godunov (1.7) and Cole-Murman (1.8).

Next let us assume $a > 0$ and consider the second order case $r = 2$ in which $Q(x; v^n)$ is a piecewise-parabolic interpolation of v^n. There are two different choices of stencil in (2.4): Taking Q in $[x_{j-1}, x_j]$ to be the parabola through $S_{j-1}^2 = \{x_{j-1}, x_j, x_{j+1}\}$ (i.e., $i(j) = j - 1$) results in the Lax-Wendroff scheme (1.9); taking Q in $[x_{j-1}, x_j]$ to be the parabola through $S_{j-2}^2 = \{x_{j-2}, x_{j-1}, x_j\}$ (i.e., $(i(j) = j - 2)$ results in the second-order upwind scheme.

We turn now to consider the application of these schemes to the step function $H(x)$

$$H(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}, H_j = \begin{cases} 0 & j \leq 0 \\ 1 & j \geq 1 \end{cases}. \tag{2.5a}$$

For the first order upwind scheme we get that

$$Q(x; H) = \begin{cases} 0 & x \leq 0 \\ x/h & 0 \leq x \leq h \\ 1 & h \leq x \end{cases}; \tag{2.5b}$$

for the Lax-Wendroff scheme

$$Q(x; H) = \begin{cases} 0 & x \leq -h \\ \frac{1}{2}\frac{x}{h}(1 + \frac{x}{h}) & -h \leq x \leq 0 \\ 1 - \frac{1}{2}(1 - \frac{x}{h})(2 - \frac{x}{h}) & 0 \leq x \leq h \\ 1 & h \leq x \end{cases}; \tag{2.5c}$$

for the second order upwind scheme we get that

$$Q(x;H) = \begin{cases} 0 & x \le 0 \\ \frac{1}{2}\frac{x}{h}(1+\frac{x}{h}) & 0 \le x \le h \\ 1+\frac{1}{2}(\frac{x}{h}-1)(2-\frac{x}{h}) & h \le x \le 2h \\ 1 & 2h \le x \end{cases} . \tag{2.5d}$$

We observe that Q in (2.5b) is a monotone function of x; consequently the numerical solution by Godunov's scheme to these data is also monotone. On the other hand Q for the second order schemes (2.5c) - (2.5d) is not a monotone function. For the Lax-Wendroff scheme Q is negative in $-h \le x \le 0$ and has a minimum of -0.125; similarly for the second order upwind scheme Q is larger than 1 in $h \le x \le 2h$ with a maximum of 1.125. This observation explains the Gibbs-like phenomenon of generating spurious oscillations in calculating discontinuous data with these second order schemes.

We say that the scheme E_h is monotonicity preserving if

$$v \text{ monotone } \Rightarrow E_h \cdot v \text{ monotone.} \tag{2.6}$$

Clearly the numerical solution of a monotonicity preserving scheme to initial data of a step-function is always monotone and therefore the discontinuity propagates without generating spurious oscillations.

Godunov has shown that the *linear* scheme (2.2) is monotonicity preserving if and only if

$$C_\ell(\nu) \ge 0, \quad -K_- \le \ell \le K_+; \tag{2.7}$$

this implies that a monotonicity-preserving scheme which is linear is necessarily only first-order accurate. It took some time to realize the Godunov's monotonicity theorem does not mean that there are no high-order accurate monotonicity preserving schemes; it only means that there are no such linear ones. Hence high-order accurate monotonicity-preserving schemes are nonlinear in an essential way.

The second-order accurate schemes mentioned above are linear because the choice of the stencil (2.4) is fixed. Let us consider now a piecewise-quadratic interpolation which is made nonlinear by an adaptive selection of the stencil in (2.4b). For the interval $[x_{j-1}, x_j]$ let us consider the two stencils $S^2_{j-2} = \{x_{j-2}, x_{j-1}, x_j\}$ and $S^2_{j-1} = \{x_{j-1}, x_j, x_{j+1}\}$, and select the one in which the interpolant is smoother. If we measure the smoothness of u by the second derivative of the corresponding parabola we select

$$i(j) = \begin{cases} j-2 & \text{if } |\frac{d^2}{dx^2}P(x;S^2_{j-2},u)| \le |\frac{d^2}{dx^2}P(x;S^2_{j-1},u)| \\ \\ j-1 & \text{otherwise} \end{cases} . \tag{2.8a}$$

When we apply this selection of stencil to the step-function $H(x)$ (2.5a) we get that for $[x_{-1}, x_0]$ we choose the stencil $S^2_{-2} = \{x_{-2}, x_{-1}, x_0\}$ for which $P(x; S^2_{-2}, H) \equiv 0$; for the interval $[x_1, x_2]$ we choose the stencil $S^2_1 = \{x_1, x_2, x_3\}$ for which $P(x; S^2_1, H) \equiv 1$. As is evident from comparing (2.5c) and (2.5d) it does not matter which stencil we

assign to $[x_0, x_1]$ since both parabolae are monotone there; with (2.8a) we select S^2_{-1} for $[x_0, x_1]$. Thus we get in (2.4)

$$Q(x;H) = \begin{cases} 0 & x \le 0 \\ \frac{1}{2}\frac{x}{h}(1+\frac{x}{h}) & 0 \le x \le h \\ 1 & h \le x \end{cases} \tag{2.8b}$$

which is a monotone function of x although it is actually a piecewise-quadratic polynomial.

The use of an adaptive stencil is the main idea behind the Essentially Non-Oscillatory (ENO) schemes to be described later in this paper. It extends to high order of accuracy in a straightforward manner: For r-th order accuracy we consider for $[x_{j-1}, x_j]$ the r stencils $S^r_{j-r}, S^r_{j-r+1}, \ldots, S^r_{j-1}$. We choose $i(j)$ in (2.4b) to be the one which minimizes

$$\left|\frac{d^r}{dx^r} P(x; S^r_i, u)\right| \quad \text{for } i = j-r, \ldots, j-1. \tag{2.9}$$

3 Total Variation Stability and TVD Schemes

An immense body of work has been done to find out whether stability of constant coefficient scheme with respect to all "frozen coefficients" associated with the problem, implies convergence in the variable coefficient case and in the nonlinear case.

In the variable coefficient case, where the numerical solution operator is linear and Lax's equivalence theorem holds, it comes out that the stability of the variable coefficient scheme depends strongly on the dissipativity of the constant coefficient one, i.e., on the particular way it damps the high-frequency components in the Fourier representation of the numerical solution.

In the nonlinear case, under assumptions of sufficient smoothness of the PDE, its solution and the functional definition of the numerical scheme, Strang proved that linear stability of the first variation of the scheme implies its convergence; we refer the reader to [13] for more details.

In the case of discontinuous solutions of nonlinear problems, linearly stable schemes are not necessarily convergent; when such a scheme fails to converge, we refer to this case as "nonlinear instability." The occurrence of a nonlinear instability is usually associated with insufficient numerical dissipation which triggers exponential growth of the high-frequency components of the numerical solution.

The following theorem states that a stronger sense of stability, namely uniform boundedness of the total variation of the numerical solution, does imply convergence to a weak solution.

Theorem 3.1. Let v_h be a numerical solution of a conservative scheme (1.5).

(i) If

$$TV(v_h(\cdot, t)) \le C \cdot TV(u_0) \tag{3.1}$$

where $TV(\ )$ denotes the total variation in x and C is a constant independent of h for $0 \le t \le T$, then any refinement sequence $h \to 0$ with $\tau = 0(h)$ has a convergent subsequence $h_j \to 0$ that converges in L_1^{loc} to a weak solution of (1.1).

(ii) If v_h is consistent with an entropy inequality which implies uniqueness of the IVP (1.1), then the scheme is convergent (i.e., all subsequences have the same limit, which is the unique entropy solution of the IVP (1.1)).

We say that the scheme E_h is Total Variation Diminishing (TVD) if

$$TV(E_h \cdot v) \leq TV(v) \tag{3.2}$$

where

$$TV(w) = \sum_j |w_{j+1} - w_j|. \tag{3.3}$$

Clearly TVD schemes satisfy (3.1) with $C = 1$ and therefore are TV stable.

In [2] we have shown that if the scheme can be written in the form

$$v_j^{n+1} = v_j^n + C^+_{j+\frac{1}{2}} \Delta_{j+\frac{1}{2}} v^n - C^-_{j-\frac{1}{2}} \Delta_{j-\frac{1}{2}} v^n \tag{3.4a}$$

where $C^{\pm}_{j+\frac{1}{2}}$ satisfy for all j

$$C^{\pm}_{j+\frac{1}{2}} \geq 0, \quad C^+_{j+\frac{1}{2}} + C^-_{j+\frac{1}{2}} \leq 1 \tag{3.4b}$$

then the scheme is TVD; here $\Delta_{i+\frac{1}{2}} v^n = v^n_{i+1} - v^n_i$. Applying this lemma to the general scheme

$$v_j^{n+1} = v_j^n - \lambda(\overline{f}_{j+\frac{1}{2}} - \overline{f}_{j-\frac{1}{2}}) \tag{3.5a}$$

$$\overline{f}_{j+\frac{1}{2}} = \frac{1}{2}(f_j + f_{j+1} - q_{j+\frac{1}{2}} \Delta_{j+\frac{1}{2}} v^n) \tag{3.5b}$$

we get that if λq satisfies

$$\lambda |\overline{a}_{j+\frac{1}{2}}| \leq \lambda q_{j+\frac{1}{2}} \leq 1 \tag{3.6a}$$

then the scheme (3.5) is TVD; here

$$\overline{a}_{j+\frac{1}{2}} = \frac{f_{j+1} - f_j}{\Delta_{j+\frac{1}{2}} v}. \tag{3.6b}$$

This shows that the Cole-Murman scheme (1.8) for which $q = |\overline{a}|$ is TVD subject to the CFL restriction $\lambda |\overline{a}_{j+\frac{1}{2}}| \leq 1$.

Using conditions (3.4b) it is possible to construct TVD schemes which are second-order accurate in the L_1-sense (see [2] and [14]). However, TVD schemes are at most second-order accurate (see [5]). In order to design higher-order accurate shock capturing schemes we introduce the notion of Essentially Non-Oscillatory (ENO) schemes.

4 ENO Schemes

In this section we describe high-order accurate Godunov-type schemes which are a generalization of Godunov's scheme (1.7) and van Leer's MUSCL scheme [10].

We start with some notations: Let $\{I_j\}$ be a partition of the real line; let $A(I)$ denote the interval-averaging (or "cell-averaging") operator

$$A(I) \cdot w = \frac{1}{|I|} \int_I w(y)dy; \tag{4.1}$$

let $\overline{w}_j = A(I_j) \cdot w$ and denote $\overline{w} = \{\overline{w}_j\}$. We denote the approximate reconstruction of $w(x)$ from its given cell-averages $\{\overline{w}_j\}$ by $R(x;\overline{w})$. To be precise, $R(x;\overline{w})$ is a piecewise-polynomial function of degree $(r-1)$, which satisfies

$$\text{(i)} \qquad R(x;\overline{w}) = w(x) + O(h^r) \quad \text{wherever } w \text{ is smooth} \tag{4.2a}$$

$$\text{(ii)} \qquad A(I_j) \cdot R(\cdot;\overline{w}) = \overline{w}_j \quad \text{(conservation)}. \tag{4.2b}$$

Finally, we define Godunov-type schemes by

$$v_j^{n+1} = A(I_j) \cdot E(\tau) \cdot R(\cdot;v^n) \equiv (\overline{E}_h \cdot v^n)_j \tag{4.3a}$$

$$v_j^0 = A(I_j)u_0; \tag{4.3b}$$

here $E(t)$ is the evolution operator of (1.1).

In the scalar case, both the cell-averaging operator $A(I_j)$ and the solution operator $E(\tau)$ are order-preserving, and consequently also total-variation diminishing (TVD); hence

$$TV(\overline{E}_h \cdot \overline{w}) \leq TV(R(\cdot;\overline{w})). \tag{4.4}$$

This shows that the total variation of the numerical solution of Godunov-type schemes is dominated by that of the reconstruction step.

We turn now to describe the recently developed essentially non-oscillatory (ENO) schemes of [5, 6], which can be made accurate to any finite order r. These are Godunov-type schemes (4.3) in which the reconstruction $R(x;\overline{w})$, in addition to relations (4.2), also satisfies

$$TV(R(\cdot;\overline{w})) \leq TV(\overline{w}) + O(h^{1+p}), \quad p > 0 \tag{4.5}$$

for any piecewise-smooth function $w(x)$. Such a reconstruction is essentially non-oscillatory in the sense that it may not have a Gibbs-like phenomenon at jump-discontinuities of $w(x)$, which involves the generation of 0(1) spurious oscillations (that are proportional to the size of the jump); it can, however, have *small* spurious oscillations which are produced in the smooth part of $w(x)$, and are usually of the size $O(h^r)$ of the reconstruction error (4.2a).

When we use an essentially non-oscillatory reconstruction in a Godunov-type scheme, it follows form (4.4) and (4.5) that the resulting scheme (4.3) is likewise essentially non-oscillatory (ENO) in the sense that for all piecewise-smooth function $w(x)$

$$TV(\overline{E}_h \cdot \overline{w}) \leq TV(\overline{w}) + O(h^{1+p}), \quad p > 0; \tag{4.6}$$

i.e., it is "almost TVD." Property (4.6) makes it reasonable to believe that the total variation of the numerical solution is uniformly bounded. We recall that by Theorem

3.1, this would imply that the scheme is convergent (at least in the sense of having convergent subsequences). This hope is supported by a very large number of numerical experiments.

Next we describe one of the techniques to obtain an ENO reconstruction. To simplify our presentation we assume that $\{I_j\}$ is a uniform partition

$$I_j = (x_{j-1}, x_j), \quad x_j = jh.$$

Given cell averages $\{\overline{w}_j\}$ of piecewise-smooth function $w(x)$, we observe that

$$h\overline{w}_j = \int_{x_{j-1}}^{x_j} w(y)dy = W(x_j) - W(x_{j-1}) \tag{4.7a}$$

where

$$W(x) = \int_{x_0}^{x} w(y)dy \tag{4.7b}$$

is the primitive function of $w(x)$. Hence we can easily compute the point values $\{W(x_j)\}$ by summation

$$W(x_i) = h \sum_{j=i_0}^{i} \overline{w}_j. \tag{4.7c}$$

Once we have computed the point values of the primitive function we use the ENO interpolation technique (2.4), (2.9) to obtain $Q(x; W)$, an r-th order piecewise-polynomial interpolation of W, i.e.,

$$Q(x; W) = P(x; S^r_{i(j)}, W) \quad \text{for} \quad x_{j-1} \le x \le x_j \tag{4.8a}$$

where $P(x; S^r_i, W)$ is the unique r-th degree polynomial which interpolates W over the stencil $S^r_i = \{x_i, x_{i+1}, \dots, x_{i+r}\}$, and $i(j)$ is chosen so that

$$|\frac{d^r}{dx^r} P(x; S^r_{i(j)}, W)| = \min_{j-r \le i \le j-1} |\frac{d^r}{dx^r} P(x; S^r_i, W)|. \tag{4.8b}$$

We define $R(x; \overline{w})$ by

$$R(x; \overline{w}) = \frac{d}{dx} Q(x; W). \tag{4.9}$$

We observe that if $w(x)$ is smooth in (x_{j-1}, x_j) then for h sufficiently small the algorithm (4.8b) will select a stencil $S^r_{i(j)}$ in which $w(x)$ is smooth. It follows then from standard interpolation theorems that

$$R(x; \overline{w}) = \frac{d}{dx} P(x; S^r_{i(j)}, W)| = \frac{d}{dx} W + O(h^r) = w(x) + O(h^r) \tag{4.10}$$

which is property (4.2a). Furthermore (4.10) holds in every interval except for those in which $w(x)$ has a discontinuity. As we have seen in the examples (2.5) and (2.8b) the Gibbs-phenomenon is associated with intervals near the discontinuity and not with the interval that contains the discontinuity. This is why the reconstruction (4.8) - (4.9) satisfies the ENO property (4.5); in [2] we show that the second-order accurate

ENO scheme is actually TVD. The conservation property (4.2b) follows directly from the definition (4.9):

$$\begin{aligned} A(I_j)R(\cdot;\overline{w}) &= \frac{1}{h}\int_{x_{j-1}}^{x_j}\frac{d}{dx}Q(x;W)dx = \frac{1}{h}[Q(x_j;W)-Q(x_{j-1};W)] \\ &= \frac{1}{h}[W(x_j)-W(x_{j-1})] = \overline{w}_j. \end{aligned} \tag{4.11}$$

The abstract scheme (4.3) can be written in the standard conservation form (1.5). To do so let us denote by $\tilde{v}(x,t)$ the solution in the small of the IVP

$$\begin{cases} (\tilde{v})_t + f(\tilde{v})_x = 0 \\ \tilde{v}(x,0) = R(x;v^n) \end{cases}, \quad 0 \le t \le \tau \tag{4.12}$$

and integrate this PDE over $I_j \times [0,\tau]$; using the divergence theorem and (4.2b) we get that v^{n+1} in (4.3) can be expressed by

$$v_j^{n+1} = v_j^n - \lambda[\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}] \tag{4.13a}$$

where

$$\hat{f}_{j+\frac{1}{2}} = \frac{1}{\tau}\int_0^\tau f(\tilde{v}(x_j,t))dt \tag{4.13b}$$

In the first-order case the scheme (4.13) is identical to Godunov's scheme and the numerical flux (4.13b) can be expressed in a closed form by (1.7b). For higher order schemes we use a numerical flux which is an appropriate approximation to (4.13b) (see [6] for more details).

We remark that the ENO schemes are related to the interpolatory schemes of Sect. 2 as follows: In the constant coefficient case a fixed choice of stencil (i.e., $i(j)-j$ = constant in (4.8a)) results in the interpolatory scheme (2.4) corresponding to the same choice of stencil.

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